Comparison between Charges on Flow and Charges on Balance in Individual-Account Pension Systems

Luis Chávez-Bedoya **Nelson Ramírez Rondán***

ESAN, Lima *Banco Central de Reserva del Perú, Lima*

Abstract

In this present article, we develop a discrete-time methodology to compare front-end load and balance fees in the accumulation phase of a defined-contribution pension fund under a system of individual accounts. Using this methodology, we study the effect of risk aversion and other relevant variables in the performance and suitability of the aforementioned types of fees. Finally, we carry out a practical application and show the results for the Peruvian Private Pension System, including indifference values between fees and certainty equivalent ratios.

Key words: Pension funds; front-end load fee; balance fee; individual accounts.

* Article received on April 17, 2015; final version approved on March 11, 2016.
The authors thank Guillermo Moloche and the Apuntes reviewer for their valuable comments, as well as the participants of the Central Reserve Bank of Peru's (Banco Central de Reserva del Perú, BCRP) research seminar, the 31st Meeting of BCRP Economists, and the 2014 Annual Congress of the Peruvian Association of Economics (Asociación Peruana de Economía, APE) for the comments and discussions that enriched this study. Any possible errors are the responsibility of the authors.
Luis Chávez-Bedoya holds a Ph.D. in Management Sciences from Northwestern University. He is currently professor of finances at the ESAN Graduate School of Business. His research has been published in *Quantitative Finance, Journal of Pension Economics and Finance, CEPAL Review, Estudios de Economía de la Universidad de Chile and Journal of Asset Management,* among other specialized journals.
Email: lchavezbedoya@esan.edu.pe
Nelson Ramírez Rondán holds a Ph.D. in Economics from the University of Wisconsin-Madison. He is currently a researcher at the BCRP Department of Economic Research. His areas of interest are theoretical econometrics and empirical macroeconomics.
Email: nramron@gmail.com

Vol. XLIII, N° 78, First Semester 2016: pages 61-88 / ISSN 0252-1865
DOI: http://dx.doi.org/10.21678/apuntes.78.835

Acronyms

AFP	Pension fund management company (Administradora de fondos de pensiones)
APE	Peruvian Association of Economics (Asociación Peruana de Economía)
BCRP	Central Reserve Bank of Peru (Banco Central de Reserva del Perú)
CRRA	Constant relative risk aversion
GBM	Geometric Brownian movement
IA	Individual account
SDE	Stochastic differential equation
SBS	Superintendence of Banking, Insurance, and AFPs (Superintendencia de Banca, Seguros y AFP)
SPP	Private Pension Pystem (Sistema Privado de Pensiones)

1. INTRODUCTION

During the final quarter of the last century, many Latin American countries reformed their pension systems, switching from public pay-as-you-go (PAYG) systems to private systems based on individual accounts (IAs).[1] According to Escrivá *et al.* (2010), these new systems constitute an attempt to adapt to the new risks and challenges faced by countries in the region, including factors such as: the vulnerability of public finances, changes in birth rates, greater life expectancy, problems of efficiency in public administration, and greater potential development of financial markets. However, a new series of reforms are now being proposed, whose fundamental objectives, discussed by Kritzer *et al.* (2011), are to increase coverage and competition in pension systems while reducing administrative costs.

Two important characteristics of IA pension systems are, on the one hand, the fact that affiliates assume the risk associated with fluctuation in the value of administrative assets; and on the other, the fact that the administrative fees (commissions) charged by pension fund management companies (administradoras de fondos de pensiones, AFPs) have a significant impact on the final balance of IAs.[2] Furthermore, as stated in James *et al.* (2001), Whitehouse (2001), and Mitchell (1998), one of the main criticisms of IA systems is their high cost, since this does nothing to encourage participation, damages the image of the systems as a whole, reduces the value of future pensions, and increases the future costs to the government of a guaranteed minimum pension.

According to Kritzer *et al.* (2011), the most common type of administrative fees in IA pension systems are: proportional charges on flow (expressed as a percentage of income or contribution), fixed charges on flow, and charges on excess returns.[3] This article analyzes

1. The most documented case is Chile. For the main aspects of this reform, see: Arrau *et al.* (1993); Diamond and Valdés-Prieto (1994); Edwards (1998); and Arenas de Mesa and Mesa-Lago (2006). In the case of Peru, a complete analysis of pension system reform and its current status is provided in Marthans and Stok (2013). Queisser (1998), Sinha (2000), Kay and Kritzer (2001), Mesa-Lago (2006), and Kritzer *et al.* (2011) are good references for the study of the reform, status, and perspective of Latin American pension systems.
2. Devesa-Carpio *et al.* (2003) argue that the levy scheme adopted in IA systems is very important because the accumulation process is exponential and directed toward long time horizons. For example, Murthi *et al.* (2001) estimate that in the United Kingdom, 40% of the value of IAs is dissipated by administrative fees, while Whitehouse (2001) finds that an annual levy of 1% of assets represents nearly 20% of the final pension value.
3. Analyses and comparison of administrative fees across different countries can be found in: James *et al.* (2001); Whitehouse (2001); Gómez-Hernández and Stewart (2008); Corvera *et al.* (2006); Tapia and Yermo (2008); Devesa-Carpio *et al.* (2003). Moreover, Sinha (2001), Masías and Sánchez (2007), and Martínez and Murcia (2008) perform a detailed analysis (notwithstanding the changes that have since been made) of administrative fees in Mexico, Peru, and Colombia, respectively.

only proportional charges on flow and income, which are the most common and important types of levy in Latin America. For Queisser (1998), charges on flow are more advantageous to AFPs during the initial phase of the system, and despite the fact that income-based commission is aligned to AFPs' objectives in terms of increasing fund profitability, they tend to be more expensive in the long run given that IAs increase in value. Meanwhile, Shah (1997) argues that charges on flow give rise to distortions and undesirable tendencies, such as engendering high AFP set-up costs, discouraging competition in the system, and generating losses for older affiliates.

In Peru, the 1992 reform of the Private Pension System (Sistema Privado de Pensiones, SPP) sought to lend the system the financial sustainability that it had previously lacked. Recently, and in the framework of SPP reform, an important point of debate has been the commissions charged by AFPs. Thus, regulation should assure the type of commission that generates the greatest terminal wealth for affiliates, which is an important consideration in the SPP.

The traditional way of comparing commission on balance with commission on flow is through a commission on equivalent risk-neutral balance, which is equal to the expected value of the funds (under both levy schemes) at the end of the period of accumulation. This approach is employed by Shah (1997), Diamond (2000), Blake and Board (2000), Whitehouse (2001), Devesa-Carpio *et al.* (2003), and Gómez-Hernández and Stewart (2008). Moloche (2012), through a model (or benchmark) that maximizes the terminal utility of an affiliate in a context of dynamic optimization, compares certain scenarios of commissions on flow and on balance, and in the empirical case of the SPP concludes that such maximization would not be apt for less risk-adverse affiliates given current levels of commission on flow.

It is in this context that this study employs two methods for comparing these commissions and, in turn, analyzes their sensitivity to changes of certain important parameters, especially affiliate risk aversion. The methods of comparison, which have been considered only from the point of view of affiliates,[4] are the ratio of expected values of terminal wealth and the difference in expected utilities of terminal wealth. In the theoretical section, a series of assumptions are made from which closed expressions can be derived to explain the behavior of the commissions under the above-mentioned methods of comparison. The most important assumptions are: the use of a geometric Brownian movement (GBM) for the quota value of the fund, and the fact that the preferences of the affiliate can be

4. A full analysis on the suitability of some of the commissions with respect to the economy could be performed as part of the general equilibrium model.

expressed in terms of the mean and the variance in terminal wealth. Subsequently, in the practical application to the SPP, a sensitivity analysis of the commissions is conducted in relation to certain fundamental variables, but by relaxing certain assumptions such as that corresponding to the function of affiliate utility. In general terms, it can be concluded that the performance of commission on balance improves as affiliate risk aversion increases, while higher growth rates on the quota value render commission on flow preferable to commission on balance in risk-neutral scenarios.

This study is structured as follows: Section 2 proposes a modeling and comparison methodology for commission on flow and commission on balance; Section 3 provides a practical application of the methodology to the SPP. Finally, Section 4 concludes, provides a number of recommendations, and proposes some extensions to the methodology.

2. METHODOLOGY

Let $i \in \mathbb{N}$ and $T \in \mathbb{N}^+$ be considered, such that $0 \leq i \leq T - 1$. The index i represents a particular month and T is the number of months remaining until the affiliate retires. It is assumed that the quota value, V, of a representative AFP pension fund in time $t \in \mathbb{R}^+$ (months) satisfies the following stochastic differential equation (SDE):

$$dV(t) = \mu V(t)\,dt + \sigma V(t)\,dB(t), V(0) = V_0, \tag{1}$$

where μ is the growth rate of the quota value by unit of time (months), σ the volatility of its monthly logarithm, V_0 the initial quota value, and the stochastic process B is a standard one-dimensional Brownian movement. The SDE in (1) is a common specification for modeling the quota value, since it is used extensively in stochastic control models for pension funds.[5]

Commissions on balance and on flow are described in detail below, using a similar structure to that in: Shah (1997), Diamond (2000), Blake and Board (2000), Whitehouse (2001), Devesa-Carpio *et al.* (2003), and Gómez-Hernández and Stewart (2008).

5. A common assumption in the literature is the use of the GBM as a process for the quota value of the fund or for the prices of the assets that make up the fund. Some examples of the use of the GBM are: Blake *et al.* (2001), Devolder *et al.* (2003), Vigna (2014), Haberman and Vigna (2002), Battocchio and Menoncin (2004), Han and Hung (2012), and Cairns *et al.* (2006), among others. In the case of the SPP, Moloche (2012) uses different BGMs to model national and international equity assets.

2.1. Commission on balance

Let $\delta > 0$ be the monthly commission on balance expressed in continuous time.[6] Moreover, in month *i* the affiliate contributes a sum $W_i > 0$ into their individual account. If the quota value, *V*, is normalized to the unit in the period *i*, then the contribution W_i is equivalent to the same number of quotas. That is, for $t \geq i$, and based on the SDE (1), the contribution made in *i* would follow the GBM below:

$$W_s^i(t) = W_i e^{(\mu - \delta - \frac{\sigma^2}{2})(t-i)+\sigma(B(t)-B(i))},\ i \leq t \leq T. \tag{2}$$

The affiliate seeks to determine the final value of their fund, $W_s(T)$, which is the sum of the final values of all contributions made per the sequence $W_T = \{W_i \mid W_i > 0, 0 \leq i \leq T-1\}$. Then,

$$W_s(T) = \sum_{i=0}^{T-1} W_s^i(T), \tag{3}$$

where the processes W_s^i in (2) are subject to the same source of uncertainty *B* given by (1). Moreover, the expectation $\mathbb{E}[W_s(T)]$, of the final value of the fund is:

$$\mathbb{E}[W_s(T)] = \sum_{i=0}^{T-1} W_i e^{(\mu-\delta)(T-i)} = e^{(\mu-\delta)T} \sum_{i=0}^{T-1} W_i e^{-(\mu-\delta)i}. \tag{4}$$

The variance in the affiliate's final fund under commission on balance is given by:

$$\mathrm{Var}(W_s(T)) = \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} W_i W_j e^{(\mu-\delta)(T-i+T-j)} \left(e^{\sigma^2(T-\max\{i,j\})} - 1\right). \tag{5}$$

See the demonstration in Annex A.

2.2. Commission on flow

Let $\alpha > 0$ be the commission on flow.[7] If the affiliate makes a contribution W_i in month *i*, the commission they would pay to the AFP (at the time of contributing) would be equal to $C_i = W_i(1 - e^{-\alpha})$. Considering that the commission C_i was invested in the fund, the affiliate's contribution adjusted for the opportunity cost of C_i can be expressed as $e^{-\alpha} W_i$. Based on

6. A constant value of δ could imply that the system has reached maturity with respect to this type of levy.
7. Also known as commission on balance, it can be levied as a percentage of an affiliate's income or contribution.

this assumption, the adjusted contribution of commission on flow in month i, W_f^i, would evolve according to the following GBM:

$$W_f^i(t) = W_i e^{-\alpha} e^{(\mu - \frac{\sigma^2}{2})(t-i) + \sigma(B(t) - B(i))}, i \leq t \leq T. \tag{6}$$

For the affiliate, it is important to calculate the value of the final fund adjusted for the opportunity cost of commission on flow per the sequence of contributions $W_T = \{W_i \mid W_i > 0, 0 \leq i \leq T-1\}$. If this final amount is denoted as $W_f(T)$, the following is obtained:

$$W_f(T) = \sum_{i=0}^{T-1} W_f^i(T). \tag{7}$$

Using (4) and the variance of the final fund (5), it is shown that the expectation and the variance of $W_f(T)$ can be calculated through the following expressions:

$$\mathbb{E}[W_f(T)] = e^{-\alpha} \sum_{i=0}^{T-1} W_i e^{\mu(T-i)} = e^{-\alpha + \mu T} \sum_{i=0}^{T-1} W_i e^{-\mu i}, \tag{8}$$

$$\mathrm{Var}(W_f(T)) = \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} W_i W_j e^{-2\alpha + \mu(T-i+T-j)} \left(e^{\sigma^2(T - \max\{i,j\})} - 1\right). \tag{9}$$

It is important to stress that $W_f(T)$ does not represent the true final value of the affiliate's fund, but one that is adjusted for the opportunity costs of the commission on flow. Moreover, the affiliate's final fund would be equal to $e^{\alpha} W_f(T)$.

2.3. Risk factors

Thus far, only one risk factor has been accounted for in the model: the returns of the fund's quota value given by (1). Randomness in the sequence of contributions has not been accounted for, despite the fact that it is generally expressed as a percentage of income which in turn, is indexed to inflation. Furthermore, in a market with stochastic rates of return, a correlation is known to exist between inflation rates and the returns on assets.[8] Thus, introducing randomness in the yields but not in the contributions could lead to false conclusions, since both processes depend on inflation. One option is to work in terms of nominal yields and by imposing a stochastic process upon contributions, but with some

8. It is well-documented that expected inflation, unexpected inflation shocks, and changes to expected inflation are negatively correlated with returns on shares. See: Fama and Schwert (1977), and Geske and Roll (1983). However, Boudoukh and Richardson (1993) find evidence which suggests that nominal long-term returns on shares are positively correlated with long-term *ex ante* and *ex post* inflation.

correlation with the quota value; however, we have opted to do so using a deterministic sequence of contributions and, to include inflation, we express and calibrate the quota value and the contributions in real terms.

2.4. Comparison of commissions on balance and on flow

Consider that an affiliate seeks to study the suitability of the commission schemes, based on terminal wealths $W_s(T)$ and $W_f(T)$ given by (3) and (7), respectively. It is important to note that upon introducing the opportunity cost of commission on flow in $W_f(T)$, the aforementioned random variables are rendered comparable in a certain sense; however, $W_f(T)$ does not represent terminal wealth but rather terminal wealth adjusted for a given opportunity cost. On the basis of this variable, the following random variables are accounted for:

$$\widetilde{W}_s(T) = W_s(T) + (1 - e^{-\alpha})W_s(T), \tag{10}$$

$$\widetilde{W}_f(T) = e^{\alpha}W_f(T). \tag{11}$$

Note that $\widetilde{W}_s(T)$ in (10) is the sum of the terminal value of the fund under commission on balance, $W_s(T)$ and the terminal values of the commissions saved compared with commission on flow, $(1 - e^{-\alpha})\, W_s(T)$, which have been reinvested in the affiliate's IA under the mechanism of voluntary contributions. Under this scheme, $\widetilde{W}_s(T)$ represents the affiliate's terminal wealth under commission on balance, assuming that the savings enabled by commissions on non-disbursed flow are reinvested under the same conditions as regular contributions on balance.[9] In turn, $\widetilde{W}_f(T)$ represents the affiliate's true terminal wealth under commission in flow. Therefore, based on definitions (10) and (11), terminal wealths under both forms of commission are compared.

The rest of the section will extend the comparison of $\widetilde{W}_s(T)$ and $\widetilde{W}_f(T)$ based on two criteria: the ratio of expected values and the difference in expected income.

2.4.1. Ratio of expected values of terminal wealth

If the comparison is made using the expectations of terminal wealth $\widetilde{W}_s(T)$ and $\widetilde{W}_f(T)$, then the following is defined:

$$\mathrm{RE}_{sf} = \frac{\mathbb{E}[\widetilde{W}_s(T)]}{\mathbb{E}[\widetilde{W}_f(T)]}. \tag{12}$$

9. For example, Moloche (2012) assumes that the savings made under commissions on flow compared with those under commissions on balance are deposited in a savings account with characteristics that are different and independent from the pension fund.

If $RE_{sf} > 1$, then commission on balance would be preferable; if $RE_{sf} < 1$, then commission on flow would be preferable. Therefore, when $RE_{sf} = 1$, the affiliate would be indifferent to both options. Note that these comparison criteria assume an affiliate who is indifferent to risk.

It is shown that the ratio of expected values of terminal wealth, RE_{sf} , is decreasing at higher rates of growth in the quota value (see Annex B). That is, RE_{sf} is a strictly decreasing function in μ and this relationship is independent of the contribution sequence W_T. The above implies that commission on flow improves compared with commission on balance when the rate of growth in the quota value increases. Below we define the equivalent commission on balance for a risk-neutral affiliate.

For a given set of parameters $N = \{T, \alpha, \mu, \sigma^2, W_T\}$ and RE_{sf}, the equivalent risk-neutral commission on balance, δ^*_N (N), is defined as the value of the commission on balance, δ, such that $RE_{sf} = 1$. Moreover, if the explicit dependence of δ^*_N (N) with respect to T, α, or both is to be denoted, it is possible to use $\delta^*_N(T)$, $\delta^*_N(\alpha)$ and $\delta^*_N(T, \alpha)$, respectively. Moreover, δ^*_N will be used when referring to equivalent commission in general.[10]

2.4.2. Expected utility of terminal wealth

Consider a risk-adverse affiliate and let $U(W)$ be their earnings when a realization of terminal wealth is equal to $W > 0$. To determine the most appropriate type of commission, the affiliate needs to compare the expected utilities of terminal wealth $\mathbb{E}[U(\widetilde{W}_s(T))]$ and $\mathbb{E}[U(\widetilde{W}_f(T))]$. However, explicit analytical expressions for these expected utilities are not generally available.

Because there are explicit formulas for the first and second moment of terminal wealths $\widetilde{W}_s(T)$ and $\widetilde{W}_f(T)$ and the objective is to find generalizable properties and/or closed formulas, then it is appropriate to consider a quadratic utility function given by:

$$U_{MV}(W) = \alpha W - bW^2, \tag{13}$$

where $\alpha > 0$ and $b > 0$ is the coefficient of risk aversion. Note that when $b = 0$, the affiliate would be risk neutral. Moreover, $\alpha = 1 + 2b\,\mathbb{E}[W]$ is fixed in a similar manner to Zhou and Li (2000). If the utility of terminal wealth, U, is like in (13), the following is obtained:

10. Note that $\delta^*_N\ (T = 1,\ \alpha) = \nu$ and $\partial_\alpha \delta^*_N\ (T,\ \alpha) > 0)$ for any scenario N; and since the ratio of expected values of terminal wealth, RE_{sf}, is decreasing at higher rates of growth in the quota value, μ, then $\partial_\mu \delta^*_N\ (T, \mu) < 0$ is obtained for $T > 1$.

$$\mathbb{E}[U_{\mathrm{MV}}(\widetilde{W}_s(T))] = \mathbb{E}[\widetilde{W}_s(T)] + b\left(2\mathbb{E}[\widetilde{W}_s(T)]^2 - \mathbb{E}[\widetilde{W}_s(T)^2]\right), \tag{14}$$

$$\mathbb{E}[U_{\mathrm{MV}}(\widetilde{W}_f(T))] = \mathbb{E}[\widetilde{W}_f(T)] + b\left(2\mathbb{E}[\widetilde{W}_f(T)]^2 - \mathbb{E}[\widetilde{W}_f(T)^2]\right), \tag{15}$$

and the expressions of $\mathbb{E}[\widetilde{W}_s(T)^2]$ and $\mathbb{E}[\widetilde{W}_f(T)^2]$ can be explicitly obtained.

For a fixed set of parameters and the quadratic utility in (13), Υ is defined as the difference between $\mathbb{E}[U_{MV}(\widetilde{W}_s(T))]$ and $\mathbb{E}[U_{MV}(\widetilde{W}_f(T))]$ given by (14) and (15), respectively. Therefore, considering Υ as a function of risk aversion, b, and the level of commission on balance, δ, the following is obtained:

$$\Upsilon(b,\delta) = \mathbb{E}[U_{\mathrm{MV}}(\widetilde{W}_s(T))] - \mathbb{E}[U_{\mathrm{MV}}(\widetilde{W}_f(T))]. \tag{16}$$

The function $\Upsilon(b, \delta)$ must be reduced as the commission on balance is increased, δ, since the commission on flow must be rendered preferable. Therefore, assumptions that assure this result are required, namely: $T > 1$, $b > 0$, and $\sigma^2 \leq \frac{1}{T}\ln(2)$ (see the formal demonstration in Annex C). Then, similarly to the definition of the equivalent risk-neutral commission on balance, the equivalent risk-adverse commission on balance is introduced as the value of δ, which makes the expected utilities of terminal wealth equal under both schemes. This measure will help in understanding the impact of risk aversion through a comparison of both types of commission.

Consider a set of parameters given by $A = N \cup \{b\}$, in which equivalent risk-adverse commission on balance, $\delta^*_A(A)$, is defined as the value of δ which makes $\mathbb{E}[U_{MV}(\widetilde{W}_s(T))]$ equal to $\mathbb{E}[U_{MV}(\widetilde{W}_f(T))]$. It is shown that commission on balance is rendered more desirable compared with commission on flow when the affiliate's risk aversion is higher (see Annex D). In other words, the foregoing implies that if the expected values of terminal wealth are identical, a risk-adverse affiliate would prefer commission on balance, and this is rendered more attractive with higher levels of risk aversion. Moreover, the commission on balance that would make both schemes indifferent (in terms of expected utility) under a given scenario is greater than the commission on balance that is equal to the expected terminal wealth under the same scenario. On this basis, under the assumptions made, the commission on balance improves its performance compared with the commission on flow when risk aversion increases. Finally, it is difficult to generalize the theoretical results in this section to other utility functions that differ from U_{MV}; however, in the numerical experiments, a function of constant relative risk aversion (CRRA) is employed and some of the theoretical results obtained in this section are empirically verified.

3. APPLICATION OF THE METHODOLOGY TO PERU'S PRIVATE PENSION SYSTEM

This section applies the proposed methodology to the SPP. This application is relevant given that the SPP is undergoing an important reform process 20 years after its creation.[11] Part of the reform entails replacing the commission on flow with commission on balance, as a result of which this study analyzes the effect of certain variables in the comparison of these commissions.

3.1. Parameters of the model

The numerical applications take into account a retirement age of 65, the current structure of commissions on flow used by the SPP, the real growth rate of income, and a scenario for the quota value, *V*, of the fund, which corresponds to the medium-risk fund (Type 2) in the SPP.

3.1.1. Calibration of the quota value GBM

To implement the methodology, it is necessary to calculate the growth parameters of the quota value, μ, and the volatility parameters of the quota value returns, σ, of the stochastic process corresponding to the quota value of the fund described by the SDE (1). It is important to mention that an SPP affiliate can change fund provided that they remain in a low-risk SPP fund (Type 1) after reaching the age of 65. Moreover, it is possible to incorporate changes of fund by the affiliate into the methodology in a framework of dynamic optimization; but in this article it is assumed that the affiliate remains in a single type of fund throughout the entire accumulation phase.

Historic logarithmic yields corrected for inflation are employed to estimate the volatility of the respective GBM.[12] For the scenario of moderate profitability (Type 2 fund), the monthly volatility of the quota value returns under a GBM is σ_M = 2.643%. Given the short history of SPP fund returns, it is to be expected that the calibrated growth rate (μ) will have a high estimation error. Therefore, and following the justifications of the "Anexo Técnico N° 1" of the Superintendencia de Banca, Seguros y AFP (SBS 2013), a real annual return equal to 5.0% is assumed for the moderate scenario (the real historical return of the Type 2 fund has been

11. Law N° 29903 provides for the primary aspects of the reform, one of which is that affiliates are to migrate to a mixed commission scheme, which includes a transitional front-loaded 10-year flow component, while from the eleventh year, the commission will be solely on balance. The reform also includes the tender mechanism for new IAs, and regulations for the incorporation of independent workers.
12. The series of historical quota value returns employed correspond to the AFP Integra, because it constitutes a suitable benchmark for the SPP; moreover, it is the only AFP that has been in existence (and which has not been merged) since the start of the available historical observations. In the case of the Type 2 fund, quota value observations from February 2, 2001 to April 30, 2014 are considered.

approximately 6.0% per annum). By way of GBM theory, it is verified that $\mu_M = r_M + 0.5\sigma_M^2$, where r_M is the expected monthly return in continuous time, and the subindex M refers to the Type 2 fund. After the appropriate transformations, $\mu_M = 0.004415$ is obtained.

3.1.2. Commission on flow

Three levels of commission on flow have been set (expressed as percentages of the affiliate's income): $f_{min} = 1.47\%$, $f_{max} = 1.69\%$ and $f_{pro} = 1.58\%$; which correspond to the minimum, maximum, and average charges on flow, respectively, under the SPP as at May 16, 2014. Because dependent workers in Peru are subject to an obligatory contribution of 10% of their income and f_i is applied to this, $\alpha = -\ln(1 - 10f)$ is obtained, with which $\alpha_{min} = 0.1590$, $\alpha_{max} = 0.185$ y $\alpha_{pro} = 0.172$.

3.1.3. Real growth in income

The monthly succession of the affiliate's real contributions (W_T) is assumed, such that $W_{i+1} = (1 + T_i)W_i$ for $i \geq 0$ and arbitrary $W_0 > 0$. The monthly factors T are calculated through the sum of the corresponding growth throughout the income curve plus an earnings component due to increased productivity. Moreover, T depends on different classifications of affiliates according to gender (F: female, M: male), education level (NU: non-university educated; U: university educated), and age. The details of the calibration of the factors are set out in SBS (2013), but it is important to mention that in the case of young affiliates, the average growth factors fluctuate between 2.5% and 3.5% per annum.

3.2. Numerical results

The parameters of the model employed in Section 3.1 are used first to determine the equivalent risk-neutral commission, δ_N^*. Then, a CRRA utility function of terminal wealth is used to evaluate the certainty equivalents generated by both types of commission and the effect that affiliate risk aversion has on them.

3.2.1. Equivalent risk-neutral commission on balance (δ_N^*)

Table 1 shows the values of this commission, δ_N^*, annualized and expressed as percentages, for certain ages, five profiles,[13] a moderate scenario (assuming a real fund return of 5.0%), and three different values of commission on flow ($\alpha_{min} = 0.1590$, $\alpha_{pro} = 0.172$, and $\alpha_{max} = 0.185$). It should be noted that δ_N^* is independent of the initial contribution $W_0 > 0$. Table 1 shows that δ_N^* is strictly increasing with age (or strictly decreasing in T) for a contribution profile and a fixed level of commission on flow - that is, the greater the age at the time of entry to the system, the more attractive the commission on flow will be.

13. The contribution profiles accounted for are: F, NU; F, U; M, NU; M, U; and E. The first four are described in Section 3.1.3, while E corresponds to a sequence of real contributions equal to $W_0 > 0$.

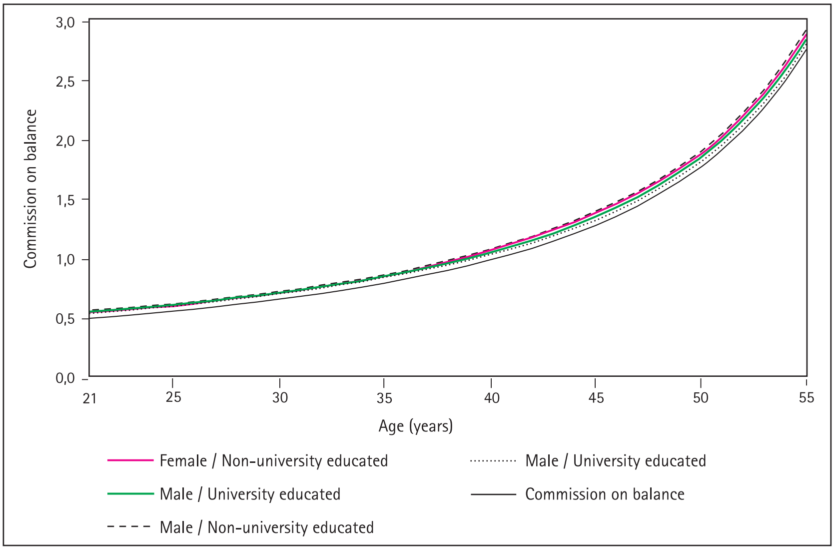
Table 1
Equivalent risk-neutral commission on balance (δ_N^*) by age and profile (in percentages and annualized)

Age	α_{min} = 0.1590					α_{pro} = 0.172					α_{max} = 0.185				
	F, NU	F, U	M, NU	M, U	E	F, NU	F, U	M, NU	M, U	E	F, NU	F, U	M, NU	M, U	E
21	0.52	0.52	0.52	0.53	0.46	0.55	0.56	0.55	0.57	0.50	0.59	0.60	0.59	0.61	0.53
22	0.53	0.54	0.53	0.54	0.48	0.57	0.58	0.57	0.58	0.51	0.60	0.61	0,61	0.62	0.54
23	0.54	0.55	0.55	0.56	0.49	0.58	0.59	0.59	0.60	0.53	0.62	0.63	0.62	0.64	0.56
24	0.56	0.57	0.56	0.57	0.51	0.60	0.61	0.60	0.61	0.54	0.64	0.65	0.64	0.65	0.58
25	0.58	0.58	0.58	0.59	0.52	0.62	0.63	0.62	0.63	0.56	0.66	0.67	0.66	0.67	0.60
26	0.59	0.60	0.60	0.60	0.54	0.63	0.64	0.64	0.65	0.58	0.68	0.69	0.68	0.69	0.61
27	0.61	0.62	0.62	0.62	0.56	0.65	0.66	0.66	0.67	0.59	0.70	0.71	0.70	0.71	0.63
28	0.63	0.64	0.64	0.64	0.57	0.68	0.68	0.68	0.69	0.61	0.72	0.73	0.73	0.73	0.65
29	0.65	0.66	0.66	0.66	0.59	0.70	0.71	0.70	0.71	0.64	0.74	0.75	0.75	0.75	0.68
30	0.67	0.68	0.68	0.68	0.61	0.72	0.73	0.73	0.73	0.66	0.77	0.78	0.78	0.78	0.70
31	0.70	0.70	0.70	0.70	0.64	0.75	0.75	0.75	0.75	0.68	0.80	0.80	0.80	0.80	0.73
32	0.72	0.73	0.73	0.72	0.66	0.77	0.78	0.78	0.78	0.71	0.82	0.83	0.83	0.83	0.75
33	0.75	0.76	0.76	0.75	0.68	0.80	0.81	0.81	0.80	0.73	0.85	0.86	0.86	0.86	0.78
34	0.78	0.78	0.79	0.78	0.71	0.83	0.84	0.84	0.83	0.76	0.89	0.89	0.90	0.89	0.81
35	0.81	0.81	0.82	0.80	0.74	0.86	0.87	0.88	0.86	0.79	0.92	0.93	0.93	0.92	0.84
36	0.84	0.85	0.85	0.84	0.77	0.90	0.91	0.91	0.89	0.82	0.96	0.96	0.97	0.95	0.88
37	0.87	0.88	0.89	0.87	0.80	0.94	0.94	0.95	0.93	0.86	1.00	1.00	1.01	0.99	0.92
38	0.91	0.92	0.93	0.90	0.84	0.98	0.98	0.99	0.97	0.90	1.04	1.05	1,06	1.03	0.96
39	0.95	0.96	0.97	0.94	0,88	1.02	1.02	1.04	1.01	0.94	1.09	1.09	1.10	1.07	1.00
40	1.00	1.00	1.01	0.98	0.92	1.07	1.07	1.08	1.05	0.98	1.14	1.14	1.16	1.12	1.05
41	1.04	1.04	1.06	1.03	0.96	1.12	1.12	1.14	1.10	1.03	1.19	1.19	1.21	1.17	1.10
42	1.09	1.09	1.11	1.08	1.01	1.17	1.17	1.19	1.15	1.08	1.25	1.25	1.27	1.23	1.16
43	1.15	1.15	1.17	1.13	1.07	1.23	1.23	1.25	1.21	1.14	1.31	1.31	1.34	1.29	1.22
44	1.21	1.21	1.23	1.19	1.12	1.30	1.29	1.32	1.27	1.20	1.38	1.38	1.41	1.36	1.28
45	1.28	1.27	1.30	1.25	1.19	1.37	1.36	1.39	1.34	1.27	1.46	1.45	1.49	1.43	1.36
46	1.35	1.34	1.38	1.32	1.26	1.45	1.44	1.48	1.42	1.35	1.55	1.53	1.57	1.51	1.44
47	1.44	1.42	1.46	1.40	1.34	1.54	1.53	1.56	1.50	1.44	1.64	1.63	1.67	1.60	1.53
48	1.53	1.51	1.55	1.49	1.43	1.64	1.62	1.67	1.60	1.53	1.75	1.73	1.78	1.70	1.64
49	1.63	1.61	1.66	1.59	1.53	1.75	1.73	1.78	1.70	1.64	1.87	1.84	1.90	1.82	1.75
50	1.75	1.73	1.78	1.70	1.65	1.88	1.85	1.91	1.82	1.77	2.00	1.98	2.03	1.94	1.88
51	1.89	1.86	1.92	1,83	1.78	2.02	1.99	2.05	1.96	1.91	2.16	2.13	2.19	2.09	2.04
52	2.04	2.01	2.07	1.98	1.93	2.19	2.16	2.22	2.12	2.07	2.33	2.30	2.37	2.26	2.21
53	2.22	2.19	2.26	2.16	2.11	2.38	2.35	2.42	2.31	2.26	2.54	2.51	2.58	2.57	2.41
54	2.44	2.40	2.47	2.37	2.32	2.61	2.58	2.65	2.53	2.49	2.79	2,75	2.83	2.70	2.66
55	2.69	2.66	2.73	2.62	2.58	2.89	2.85	2.93	2.80	2.76	3.08	3.04	3.12	2.99	2.95

Notes
- F: female
- M: male
- NU: non-university educated
- U: university educated
- E: case of equal contributions
- Risk-neutral commission on balance, δ_N^*, in percentages for different ages and combinations of gender and university education under the moderate scenario (μ_M 0.0044 and σ_M = 2.643% per month).
- The case of equal contributions has also been included in the table (E).
- A retirement age of 65 has been factored in, α_{min} = 0.590, α_{pro} = 0,172 and α_{max} = 0.185 (these three values correspond to commissions on flow of 1.49%, 1.58% and 1.69% on income, respectively, under a contribution rate of 10%).

Considering a 30-year-old affiliate pertaining to the M, NU profile, δ_N^*= 0.68% per annum is obtained for α_{min} = 0.1590, δ_N^*= 0.73% for α_{pro} = 0.172 and δ_N^*= 0.78% for α_{max} = 0.185. For example, this implies that if a commission on flow equal to α_{min} is used, then a charge on flow of less than 0.68% would render commission on flow preferable for all affiliates over the age of 30 and pertaining to the M, NU profile who enter the system. Note that δ_N^*> 0.52% for all scenarios in Table 1 (without accounting for profile E). This value corresponds to a 21-year-old affiliate with a F, NU profile and a commission on flow equal to α_{min}. That is, a level of commission on flow of less than or equal to 0.52% would render this commission preferable (for a risk-neutral affiliate) under all contribution profiles (without E) and scenarios included in the study. The equal contributions profile (E) benefits the commission on flow, since the values of δ_N^* generated are lower than those corresponding to other profiles. This is based on the fact that the rise in income can be considered as an increase in the growth rate μ. Finally, and by way of example, Graph 1 shows equivalent risk-neutral commission on balance, δ_N^*, as a function of age and contribution profile, but only for average commission on flow, α_{prom}.

Graph 1
Risk-neutral commission on balance δ_N^* by age (in percentages)



Notes

- Risk-neutral commission on balance, δ_N^*, in percentages and annualized for different combinations of age and gender/university education under the moderate scenario (μ_M 0.0044 and σ_M = 2.643% per month).
- The case of equal contributions (E) has been included.
- α = 0.172 has been assumed (corresponding to a charge on flow equal to 1.58% of income under a constant contribution rate of 10% of income) and a retirement age of 65.

3.2.2. Comparison of certainty equivalents $(\Delta\mathrm{CE}_{sf})$

This section will empirically study the following ratio:

$$\Delta\mathrm{CE}_{sf} = \frac{\mathrm{CE}[\widetilde{W}_s(T)]}{\mathrm{CE}[\widetilde{W}_f(T)]} - 1, \tag{17}$$

where $\mathrm{CE}[\widetilde{W}_s(T)]$ and $\mathrm{CE}[\widetilde{W}_f(T)]$ are the certainty equivalents of $\widetilde{W}_s(T)$ and $\widetilde{W}_f(T)$, assuming an arbitrary utility function U of terminal wealth - that is, it satisfies both $\mathbb{E}[U(\widetilde{W}_s(T))] = U(\mathrm{CE}[\widetilde{W}_s(T)])$ and $\mathbb{E}[U(\widetilde{W}_f(T))] = U(\mathrm{CE}[\widetilde{W}_f(T)])$. The value of $\Delta\mathrm{CE}_{sf}$ under the quadratic utility function given by (13) has an explicit analytical expression because the expressions for the means and variances of $\widetilde{W}_s(T)$ and $\widetilde{W}_f(T)$ are available and can be used in (17). But, in this case, $\Delta\mathrm{CE}_{sf}$ will depend on W_0 even if it is assumed that W_T, as in Section 3.1.3. In consequence, this fact complicates the study of the behavior of $\Delta\mathrm{CE}_{sf}$ because of changes in the coefficient of risk aversion, *b*, of the quadratic utility function given by (13).

An option for eliminating the dependence of W_0 is to employ a utility function, such that $\Delta\mathrm{CE}_{sf}$ will be independent of W_0 when W_T is assumed, as in Section 3.1.3. If a CRRA utility function is considered such that:

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \tag{18}$$

where $W > 0$ is the terminal wealth and $\Upsilon > 0$ is the risk aversion coefficient, then $\Delta\mathrm{CE}_{sf}$ will not depend on W_0. This fact will be important in separating the effect of risk aversion on $\Delta\mathrm{CE}_{sf}$ from that caused by the initial contribution W_0. It is important to mention that closed expressions for $\mathrm{CE}[\widetilde{W}_s(T)]$ and $\mathrm{CE}[\widetilde{W}_f(T)]$ are not available for the utility (18), with which simulation would have to be used to obtain an estimator of $\Delta\mathrm{CE}_{sf}$ in (17). Moreover, in the literature on pension fund management, the use of a CRRA utility is more common and appropriate, as in (18), than the quadratic given by (13).

Table 2 presents the estimated values of $\Delta\mathrm{CE}_{sf}$ for different ages, the F, NU contribution profile (since the results for other profiles in Section 3.1.3 are very similar, they are not reported), and under the moderate profitability scenario described in Section 3.1.1. Following Poterba *et al.* (2005), three different values of affiliate risk aversion are accounted for: $\Upsilon = 1$ for a low degree; and in this case $U(W) = \ln(W)$, $\Upsilon = 4$ for a moderate degree; and $\Upsilon = 8$ for a high degree. The level of commission on balance, δ, is set at three levels: 0,5%, 1,0% and 1,5% per annum, while the level of commission on flow is set at $\alpha_{pro} = 0.172$ (current average value of the SPP). The number of sample paths of wealth used to estimate ΔCE_{sf} is determined using the sequential procedure of Kelton and Law (2000) with a relative error of 0.0001 and a confidence level of 99%. Moreover, Graph 2 outlines all of the information contained in Table 2.

Table 2

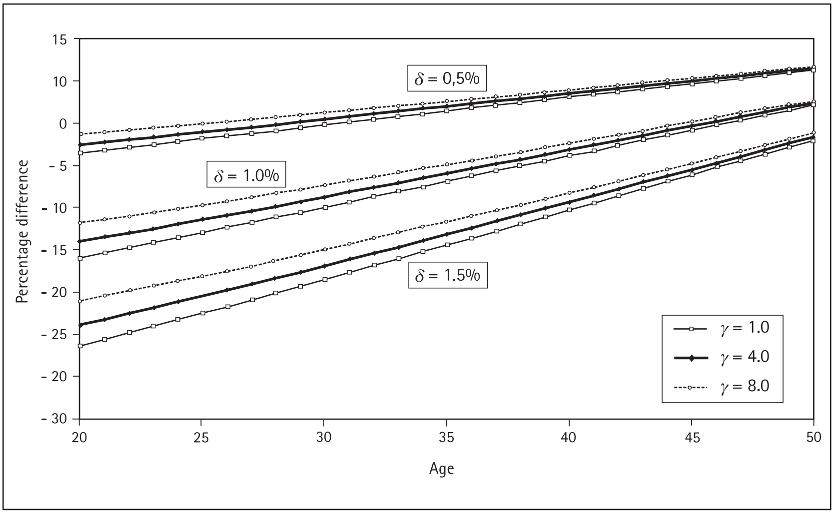
Estimated certainty equivalent values, ΔCE_{sf} by age (in percentages)

Age	δ = 0.50%			δ = 1.00%			δ = 1.50%		
	γ = 1	γ = 4	γ = 8	γ = 1	γ = 4	γ = 8	γ = 1	γ = 4	γ = 8
20	1.38	2.40	3.63	- 10.89	- 9.07	- 6.82	- 21.32	- 18.94	- 16.01
21	1.71	2.70	3.92	- 10.31	- 8.56	- 6.47	- 20.59	- 18.30	- 15.40
22	2.04	2.99	4.15	- 9.74	- 8.06	- 6.05	- 19.85	- 17.61	- 14.80
23	2.38	3.29	4.39	- 9.16	- 7.53	- 5.59	- 19.08	- 16.92	- 14.31
24	2.71	3.59	4.63	- 8.56	- 7.00	- 5.14	- 18.31	- 16.24	- 13.61
25	3.05	3.90	4.89	- 7.98	- 6.47	- 4.64	- 17.54	- 15.54	- 13.11
26	3.39	4.20	5.14	- 7.38	- 5.94	- 4.21	- 16.76	- 14.83	- 12.58
27	3.74	4.50	5.41	- 6.78	- 5.41	- 3.75	- 15.97	- 14.13	- 11.92
28	4.08	4.80	5.68	- 6.17	- 4.87	- 3.24	- 15.16	- 13.41	- 11.25
29	4.42	5.11	5.97	- 5.56	- 4.32	- 2.86	- 14.36	- 12.67	- 10.63
30	4.76	5.42	6.23	- 4.96	- 3.78	- 2.32	- 13.55	- 11.94	- 9.99
31	5.11	5.73	6.47	- 4.35	- 3.22	- 1.85	- 12.73	- 11.19	- 9.34
32	5.45	6.04	6.74	- 3.74	- 2.67	- 1.38	- 11.92	- 10.45	- 8.69
33	5.79	6.34	7.03	- 3.14	- 2.12	- 0.87	- 11.09	- 9.71	- 8.00
34	6.13	6.65	7.28	- 2.53	- 1.56	- 0.40	- 10.27	- 8.97	- 7.33
35	6.46	6.96	7.56	- 1.92	- 1.01	0.10	- 9.45	- 8.20	- 6.70
36	6.80	7.26	7.82	- 1.31	- 0.46	0.57	- 8.62	- 7.45	- 6.01
37	7.13	7.57	8.09	- 0.70	0.10	1.08	- 7.79	- 6.69	- 5.33
38	7.47	7.87	8.37	- 0.09	0.66	1.57	- 6.96	- 5.92	- 4.63
39	7.80	8.18	8.64	0.51	1.21	2.09	- 6.13	- 5.16	- 3.94
40	8.13	8.48	8.92	1.12	1.77	2.56	- 5.29	- 4.39	- 3.25
41	8.46	8.78	9.17	1.72	2.32	3.08	- 4.47	- 3.62	- 2.57
42	8.78	9.08	9.45	2.32	2.88	3.58	- 3.63	- 2.85	- 1.88
43	9.10	9.38	9.72	2.92	3.44	4.07	- 2.80	- 2.07	- 1.16
44	9.42	9.68	9.99	3.52	3.99	4.59	- 1.97	- 1.30	- 0.46
45	9.74	9.97	10.26	4.11	4.54	5.09	- 1.14	- 0.52	0.25
46	10.06	10.27	10.53	4.70	5.09	5.58	- 0.31	0.25	0.95
47	10.37	10.56	10.79	5.29	5.65	6.09	0.52	1.03	1.66
48	10.69	10.85	11.06	5.88	6.20	6.60	1.36	1.81	2.38
49	11.00	11.15	11.34	6.47	6.75	7.11	2.20	2.60	3.11
50	11.31	11.44	11.60	7.06	7.31	7.63	3.03	3.39	3.84

Notes

- Estimated values of $\Delta CE_{sf} = CE[\widetilde{W}_s(T)] / CE[\widetilde{W}_f(T)] - 1$ in percentages for different ages, risk aversion coefficient values (γ = 1, 4, 8), F, SU contribution profile as per Section 3.1.3 and the moderate scenario (μ_M = 0.0044 and σ_M = 2.643% per month).
- δ = 0.5%, 1.0%, 1.5% per annum α = 0,172 has been assumed (corresponding to a charge on flow equal to 1.58% of income under a constant contribution rate of 10% of income) and a retirement age of 65.

Graph 2
Percentage difference in CE (moderate scenario; in percentages)



Notes

- Estimated values of $\Delta CE_{sf} = CE[\widetilde{W}_s(T)]/CE[\widetilde{W}_f(T)] - 1$ in percentages, for different ages, risk aversion coefficient values ($\gamma = 1, 4, 8$), and under the moderate scenario ($\mu_M = 0.0044$ and $\sigma_M = 2.643\%$ per month) with F, NU contribution profile as per Section 3.1.3.
- $\delta = 0{,}5\%, 1{,}0\%, 1{,}5\%$ per annum $\alpha = 0.172$ has been assumed (corresponding to a charge on flow equal to 1.58% of income under a constant contribution rate of 10% of income) and a retirement age of 65.

From the information provided, it can be observed that ΔCE_{sf} is an increasing function in the degree of risk aversion, Υ, for a fixed age and level of commission, which empirically corroborates the fact that with greater risk aversion, commission on balance is improved compared with commission on flow. However, $\Delta CE_{sf} > 0$ is not obtained for all ages, since δ is fixed at certain fixed values instead of working with risk-neutral commission on balance, δ_N^*, corresponding to each age. Moreover, when δ decreases, commission on balance is rendered more attractive, since the ΔCE_{sf} curves for a fixed Υ go up. It is also observed that the ΔCE_{sf} slope tends to be positive, remaining constant with age and decreasing slightly as the affiliate's degree of risk aversion increases. It is important to note that $\delta = 0.5\%$ guarantees that $\Delta CE_{sf} > 0$ for all ages and levels of risk aversion accounted for. Moreover, when $\delta = 1\%$, the indifference age between commissions fluctuates between 35 ($\Upsilon = 8$) and 38 ($\Upsilon = 1$). When $\delta = 0.5\%$, the indifference age is around 45 ($\Upsilon = 8$) and 47 ($\Upsilon = 1$). It should be recalled that affiliates who are below the difference age will prefer commission on flow. If $\delta = 1\%$ and the commission on flow is 1.58% of income (SPP average as at May 2014), a 20-year-old affiliate would experience a percentage loss in certainty equivalent in the range of 7% ($\Upsilon = 8$) to 11% ($\Upsilon = 1$); while if the commission is $\delta = 1.5\%$, the loss would be in the range of 16% ($\Upsilon = 8$) to 21% ($\Upsilon = 1$).

4. CONCLUSIONS

This study develops a discrete-time method that enables comparison of commissions on flow with commissions on balance in IA pension systems. The comparison methods employed are the ratio of expected values of terminal wealth and the difference in expected utilities of terminal wealth. In many cases, very general results are obtained with respect to the relative performance of the commission schemes and these are achieved without the need to assume a particular pattern in the sequence of contributions. Moreover, formulas and/or expressions are provided in order to determine the most appropriate type of commission for each affiliate. A quadratic utility function is employed to demonstrate theoretically that, in general, increases in risk aversion improve the performance of the commission on balance compared with commission on flow, a result which is in keeping with the empirical conclusion of Moloche (2012).

On the basis of the theoretical analysis carried out and its practical application to the SPP, it can be stated that commission on balance equal to 0.5% per annum would render commission on balance preferable to commission on flow across almost all scenarios included in the study, based on commission on flow equal to 15.80% of contributions (SPP average as at May 2014). At this level of commission on flow and when commission on balance is equal to 1% per annum, affiliates below the age of 37 (approximately) will prefer the charge on flow; but when commission is 1.5% per annum, the indifference age increases to 45 (approximately). Moreover, an affiliate who enters the system at 20 years of age under a commission on balance scheme equal to 1% per annum could lose between 7% and 11% in terminal wealth certainty equivalent, in comparison with commission on flow. If the rate of commission were 1.5% per annum, the percentage loss would be between 16% and 21%. It is important to state that the lower and upper values in the ranges correspond to a high and low level of risk aversion, respectively.

It is possible to make many refinements to the methodology and the comparison methods. For example, variable commissions on balance according to the evolution of equivalent commissions from the perspective of the AFP, and commission on results that are strictly dependent on the stock exchange, more sophisticated stochastic processes for the quota value, and the relevant economic variables could be considered. Moreover, it would be interesting to contrast the schemes by using optimal policies that allow changes in the level of fund risk and returns according to the age of the affiliate and the size of the fund, in a context of discrete-time stochastic optimization. In these terms, the study by

Moloche (2012) could be built upon for different ages and reinvestment rates of saved commissions on flow. In addition, it is possible to work under the assumption of market completion and provide expressions to determine indifference values between fees by using valuation in the absence of arbitrage opportunities. But such extensions are beyond the scope of this study.

ANNEXES

Annex A

Demonstration of the terminal wealth variable

For the calculation of Var $(W_s(T))$, it is necessary to calculate the variance of $W_s^i(T)$ and the covariance between $W_s^i(T)$ and $W_s^j(T)$. Because $W_s^i(T)$ has a log-normal distribution, then:

$$\mathrm{Var}\,(W_s^i(T)) = W_i^2\, e^{2\,(\mu-\delta)(T-i)} \left(e^{\sigma^2 (T-i)} - 1\right). \tag{19}$$

Therefore, it is to be verified that for $j > i$ and $W_j > 0$:

$$\mathrm{Cov}\,(W_s^i(T), W_s^j(T)) = \sigma_{W_s^i(T), W_s^j(T)} = \mathrm{Var}\,(W_s^j(T)) \frac{W_i}{W_j}\, e^{(\mu-\delta)\,(j-i)}, \tag{20}$$

where the GBM W_s^i is defined in (2).

From the properties of W_s^i, the following is obtained:

$$\sigma_{W_s^i(T), W_s^j(T)} = W_i W_j\, e^{(\mu-\delta-\frac{\sigma^2}{2})(T-i+T-j)} \mathrm{Cov}\left(e^{\sigma(B(T)-B(i))}, e^{\sigma(B(T)-B(j))}\right) \tag{21}$$

$$= W_i W_j\, e^{(\mu-\delta-\frac{\sigma^2}{2})(T-i+T-j)}\; \mathrm{Cov}\left(e^{\sigma(B(T)-B(j)+B(j)-B(i))}, e^{\sigma(B(T)-B(j))}\right) \tag{22}$$

$$= W_i W_j\, e^{(\mu-\delta-\frac{\sigma^2}{2})(T-i+T-j)}\; \mathbb{E}\left[e^{\sigma(B(j)-B(i))}\right] \mathrm{Var}\left(e^{\sigma(B(T)-B(j))}\right) \tag{23}$$

$$= W_i W_j\; e^{(\mu-\delta-\frac{\sigma^2}{2})\,(T-i+T-j)}\, e^{\frac{\sigma^2}{2}\,(j-i)}\, e^{\sigma^2\,(T-j)} \left(e^{\sigma^2\,(T-j)} - 1\right) \tag{24}$$

$$= W_j^2\, e^{2\,(\mu-\delta)\,(T-j)} \left(e^{\sigma^2\,(T-j)} - 1\right) \frac{W_i}{W_j}\; e^{(\mu-\delta)\,(j-i)} \tag{25}$$

$$= \mathrm{Var}\,(W_s^j\,(T)) \frac{W_i}{W_j}\; e^{(\mu-\delta)\,(j-i)}. \tag{26}$$

To obtain (23), the fact that $B\,(T) - B\,(j)$ is independent of $B\,(j) - B\,(i)$ is utilized.

Formulas (19) and (20) generate the following expression for the covariance of the final values of the contributions i and j for all $0 \le i \le T-1$ and $0 \le j \le T-1$:

$$\mathrm{Cov}\,(W_s^i(T), W_s^j(T)) = W_i\, W_j\, e^{(\mu-\delta)(T-i+T-j)} \left(e^{\sigma^2(T-\max\{i,j\})} - 1\right). \tag{27}$$

By using (27), the variance of $W_s\,(T)$ can be calculated through:

$$\mathrm{Var}\left(W_s(T)\right) = \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} W_i\,W_j\,e^{(\mu-\delta)(T-i+T-j)}\left(e^{\sigma^2(T-\max\{i,j\})}-1\right). \quad (28)$$

ANNEX B

Demonstration that the derivative of $\mathrm{RE}_{sf}(\mu)$ with respect to μ is negative

Let $v = \ln(2 - e^{-\alpha})$. For the case of contributions according to sequence W_T and when $T > 1$, it can be expressed that RE_{sf} in (12) as follows:

$$\mathrm{RE}_{sf} = e^{\nu-\delta T}\frac{W_0 + W_1 e^{-(\mu-\delta)} + \cdots + W_{T-1}e^{-(\mu-\delta)(T-1)}}{W_0 + W_1 e^{-\mu} + \cdots + W_{T-1}e^{-\mu(T-1)}}. \quad (29)$$

The following is obtained, based on the expression given in (29):

$$\mathrm{RE}_{sf}(\mu) = e^{\nu-\delta T}\frac{W_0 + \sum_{i=1}^{T-1} W_i\,e^{-(\mu-\delta)\,i}}{W_0 + \sum_{i=1}^{T-1} W_i\,e^{-\mu i}}. \quad (30)$$

The partial derivative of $\mathrm{RE}_{sf}(\mu)$ with respect to μ is:

$$\frac{\partial \mathrm{RE}_{sf}}{\partial \mu} = e^{\nu-\delta T}\frac{\left(-\sum_{i=1}^{T-1} W_i i e^{-(\mu-\delta)\,i}\right)\left(W_0 + \sum_{i=1}^{T-1} W_i e^{-\mu i}\right)}{\left(W_0 + \sum_{i=1}^{T-1} W_i e^{-\mu i}\right)^2} + e^{\nu-\delta T}\frac{\left(W_0 + \sum_{i=1}^{T-1} W_i\,e^{-(\mu-\delta)\,i}\right)\left(\sum_{i=1}^{T-1} W_i\,i e^{-\mu i}\right)}{\left(W_0 + \sum_{i=1}^{T-1} W_i\,e^{-\mu i}\right)^2} \quad (31)$$

$$= \frac{e^{\nu-\delta T}}{\left(W_0 + \sum_{i=1}^{T-1} W_i e^{-\mu i}\right)^2}\left\{W_0 \sum_{i=1}^{T-1} W_i\,i\,e^{-\mu i}\left(1 - e^{\delta i}\right) + \sum_{i=1}^{T-1} W_i i e^{-\mu i}\sum_{i=1}^{T-1} W_i e^{-(\mu-\delta)i} - \sum_{i=1}^{T-1} W_i i e^{-(\mu-\delta)i}\sum_{i=1}^{T-1} W_i e^{-\mu i}\right\}. \quad (32)$$

Because $\delta > 0$ and W_i for all i, it is clear that $W_0 \sum_{i=1}^{T-1} W_i i e^{-\mu i}\left(1-e^{\delta i}\right) < 0$ for $T > 1$. On the basis of which, to demonstrate that $\frac{\partial \mathrm{RE}_{sf}}{\partial \mu} < 0$, it would only remain to be verified that:

$$\sum_{i=1}^{T-1} W_i i e^{-(\mu-\delta)i} \sum_{i=1}^{T-1} W_i e^{-\mu i} \geq \sum_{i=1}^{T-1} W_i i e^{-\mu i} \sum_{i=1}^{T-1} W_i e^{-(\mu-\delta)i}. \quad (33)$$

Proceeding by induction, for $T = 2$ in (33) the following is obtained:

$$W_1 e^{-(\mu-\delta)} W_1 e^{-\mu} \geq W_1 e^{-\mu} W_1 e^{-(\mu-\delta)}. \quad (34)$$

Assuming that (33) is fulfilled, it must be demonstrated that:

$$\sum_{i=1}^{T} W_i i e^{-(\mu-\delta)i} \sum_{i=1}^{T} W_i e^{-\mu i} \geq \sum_{i=1}^{T} W_i i e^{-\mu i} \sum_{i=1}^{T} W_i e^{-(\mu-\delta)i}. \quad (35)$$

The inequality (35) is equivalent to:

$$\sum_{i=1}^{T-1} W_i i e^{-(\mu-\delta)i} \sum_{i=1}^{T-1} W_i e^{-\mu i} + W_T e^{-\mu T} \sum_{i=1}^{T-1} W_i i e^{-(\mu-\delta)i} +$$

$$W_T T e^{-(\mu-\delta)T} \sum_{i=1}^{T-1} W_i e^{-\mu i} \geq \sum_{i=1}^{T-1} W_i i e^{-\mu i} \sum_{i=1}^{T-1} W_i e^{-(\mu-\delta)i} +$$

$$W_T e^{-(\mu-\delta)T} \sum_{i=1}^{T-1} W_i i e^{-\mu i} + W_T T e^{-\mu T} \sum_{i=1}^{T-1} W_i e^{-(\mu-\delta)i}. \quad (36)$$

Because it is assumed that (33) and $W_T > 0$, the inequality (36) implies:

$$\sum_{i=1}^{T-1} W_i i e^{-(\mu-\delta)i} + T e^{\delta T} \sum_{i=1}^{T-1} W_i e^{-\mu i} \geq e^{\delta T} \sum_{i=1}^{T-1} W_i i e^{-\mu i} + \quad (37)$$

$$T \sum_{i=1}^{T-1} W_i e^{-(\mu-\delta)i},$$

which in turn is equivalent to:

$$\sum_{i=1}^{T-1} W_i e^{-\mu i}(T-i)\left(e^{\delta T} - e^{\delta i}\right) \geq 0. \quad (38)$$

To conclude, it is observed that the inequality (38) is fulfilled, since $T - i \geq 1$, $W_i > 0$ and $\delta > 0$.

Annex C

Monotonicity of $\Upsilon(b, \delta)$ with respect to δ

From the definition of Υ given by (16), the following is obtained:

$$\partial_\delta \Upsilon(b,\delta) = \partial_\delta \mathbb{E}[\widetilde{W}_s(T)] + b \times \partial_\delta \left(2\mathbb{E}[\widetilde{W}_s(T)]^2 - \mathbb{E}[\widetilde{W}_s(T)^2]\right). \tag{39}$$

Since $\partial_\delta \mathbb{E}[\widetilde{W}_s(T)] < 0$ for scenario A and it is assumed that $b > 0$, only the conditions for $\partial_\delta(2\,\mathbb{E}[\widetilde{W}_s(T)]^2 - \mathbb{E}[\widetilde{W}_s(T)^2]) \leq 0$ must be established. Using the expressions for $\mathbb{E}[\widetilde{W}_s(T)]^2$ and $\mathbb{E}[\widetilde{W}_s(T)^2]$, it is verified that:

$$\partial_\delta \mathbb{E}[\widetilde{W}_s(T)]^2 = -e^{2\nu} \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} (2T-i-j)\, W_i W_j e^{(\mu-\delta)(2T-i-j)}, \tag{40}$$

$$\partial_\delta \mathbb{E}[\widetilde{W}_s(T)]^2 = -e^{2\nu} \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} (2T-i-j)$$
$$W_i W_j e^{(\mu-\delta)(2T-i-j) + \sigma^2 (T - \max\{i,j\})}. \tag{41}$$

From the results before and after some simplifications, the following is obtained:

$$e^{2\nu} \times \partial_\delta (2\mathbb{E}[\widetilde{W}_s(T)]^2 - \mathbb{E}[\widetilde{W}_s(T)^2]) = -2 \sum_{i=0}^{T-1} (T-i) W_i^2 e^{2(\mu-\delta)(T-i)}$$
$$\left(2 - e^{\sigma^2 (T-i)}\right) - \sum_{i=0, i\neq j}^{T-1} \sum_{j=0}^{T-1} (2T-i-j) W_i W_j e^{(\mu-\delta)(2T-i-j)}$$
$$\left(2 - e^{\sigma^2(T-\max\{i,j\})}\right). \tag{42}$$

Finally, (42) will be less than or equal to zero if $2 - e^{\sigma^2 (T-i)} \geq 0$ for all i, which is fulfilled when $2 - e^{\sigma^2 T} \geq 0$.

Annex D

For the case of high risk aversion, commission on balance is preferable to commission on flow

First, the following ratios are defined:

$$\mathrm{H}_s = \frac{\mathbb{E}[\widetilde{W}_s(T)]}{\sqrt{\mathrm{Var}\,(\widetilde{W}_s(T))}}, \;\; \mathrm{y} \;\; \mathrm{H}_f = \frac{\mathbb{E}[\widetilde{W}_f(T)]}{\sqrt{\mathrm{Var}\,(\widetilde{W}_f(T))}}. \tag{43}$$

If δ is fixed at the level given by δ^*_{N} and the fact that $\mathrm{H}_s > \mathrm{H}_f$ is used (the demonstration is available upon request), then $\mathbb{E}[W_f(T)] = \mathbb{E}[W_s(T)]$, $\mathbb{E}[W_f(T)^2] > \mathbb{E}[W_s(T)^2]$, and:

$$\Upsilon(b, \delta^*_{\mathrm{N}}) = \mathbb{E}[U(\widetilde{W}_s(T))] - \mathbb{E}[U(\widetilde{W}_f(T))] = b\left(\mathbb{E}[\widetilde{W}_f(T)^2] - \mathbb{E}[\widetilde{W}_s(T)^2]\right) > 0. \tag{44}$$

are obtained.

Since the right side of (44) is positive and increasing in b, Υ (b, δ^*_{N}) > 0 as well as $\partial_b \Upsilon(b, \delta^*_{\mathrm{N}}) > 0$ are obtained. Then, for $\delta > 0$, the function $\Lambda(\delta)$ is defined as:

$$\Lambda(\delta) = 2\mathbb{E}[\widetilde{W}_s(T)]^2 - \mathbb{E}[\widetilde{W}_s(T)^2] - (2\mathbb{E}[\widetilde{W}_f(T)]^2 - \mathbb{E}[\widetilde{W}_f(T)^2]). \tag{45}$$

From previous demonstrations it is known that if $\sigma^2 \leq \frac{1}{T}\ln(2)$, then $\partial_\delta \Lambda(\delta) < 0$. Moreover, $\delta^*_\infty > \delta^*_{\mathrm{N}}$ will exist, such that $\Lambda(\delta^*_\infty) = 0$. For any $b > 0$, the following is obtained:

$$\Upsilon(b, \delta^*_\infty) = \mathbb{E}[\widetilde{W}_s(T)] - \mathbb{E}[\widetilde{W}_f(T)] = K. \tag{46}$$

Note that $K < 0$ is independent of b. If two arbitrary values of the risk aversion coefficient, $b_1 > 0$ and $b_2 > 0$, are taken, such that $b_1 > b_2$ and $\delta \in (\delta^*_{\mathrm{N}}, \delta^*_\infty)$, then it can be affirmed that $\Upsilon(b_1, \delta) > \Upsilon(b_2, \delta)$ with δ inside the previously established interval, and also that $\delta^*_{\mathrm{A}}(b_1)$ y $\delta^*_{\mathrm{A}}(b_2)$. Note that both, $\delta^*_{\mathrm{A}}(b_1)$ and $\delta^*_{\mathrm{A}}(b_2)$, exist because $\Upsilon(b, \delta^*_{\mathrm{N}}) > 0$, $\Upsilon(b, \delta^*_\infty) < 0$, and $\partial_\delta \Upsilon(b, \delta) < 0$ for any $b > 0$.

REFERENCES

ARENAS DE MESA, Alberto and Carmelo MESA-LAGO

2006 "The Structural Pension Reform in Chile: Effects, Comparisons with other Latin American Reforms, and Lessons." In: *Oxford Review of Economic Policy,* vol. 22, no. , pp. 149-167.

ARRAU, Patricio; Salvador VALDÉS-PRIETO and Klaus SCHMIDT-HEBBEL

1993 "Privately Managed Pension Systems: Design Issues and the Chilean Experience" [mimeo.]. Technical report. Washington D.C.: World Bank.

BATTOCCHIO, Paolo and Francesco MENONCIN

2004 "Optimal Pension Management in a Stochastic Framework." In: *Insurance: Mathematics and Economics,* vol. 34, no. 1, pp. 79-95.

BLAKE, David and John BOARD

2000 "Measuring Value Added in the Pensions Industry." In: *Geneva Papers on Risk and Insurance-Issues and Practice,* vol. 25, no. 4, pp. 539-567.

BLAKE, David; Andrew J. G. CAIRNS and Kevin DOWD

2001 "Pensionmetrics: Stochastic Pension Plan Design and Value-at-risk During the Accumulation Phase." In: I*nsurance: Mathematics and Economics,* vol. 29, no. 2, pp. 187-215.

BOUDOUKH, Jacob and Matthew RICHARDSON

1993 "Stock Returns and Inflation: A Long-horizon Perspective." In: *The American Economic Review,* vol. 83, no. 5, pp. 1346-1355.

CAIRNS, Andrew J. G.; David BLAKE and Kevin DOWD

2006 "Stochastic Lifestyling: Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans." In: *Journal of Economic Dynamics and Control,* vol. 30, no. 5, pp. 843-877.

CORVERA, F.; J. LARTIGUE and D. MADERO

2006 "Análisis comparativo de las comisiones por administración de los fondos de pensiones en los países de América Latina" [mimeo.] Mexico.

DEVESA-CARPIO, José Enrique; R. RODRÍGUEZ-BARRERA and Carlos VIDAL-MELIÁ

2003 "Medición y comparación internacional de los costes de administración para el afiliado en las cuentas individuales de capitalización." In: *Revista Española de Financiación y Contabilidad,* vol. XXXII, no. 116, pp. 95-144.

DEVOLDER, Pierre; Manuela BOSCH PRINCEP and Inmaculada DOMÍNGUEZ FABIÁN

2003 "Stochastic Optimal Control of Annuity Contracts." In: *Insurance: Mathematics and Economics,* vol. 33, no. 2, pp. 227-238.

DIAMOND, Peter

2000 "Administrative Costs and Equilibrium Charges with Individual Accounts." In: SHOVEN, J. (ed.). *Administrative Aspects of Investment-based Social Security Reform.* Chicago: University of Chicago Press, pp. 137-172.

DIAMOND, Peter and Salvador VALDÉS-PRIETO

1994 "Social Security Reforms." In: BOSWORTH, B. P.; R. DORNBUSCH and R. LABÁN (eds.). *The Chilean Economy: Policy Lessons and Challenges.* Washington D.C.: The Brookings Institution, pp. 257-328.

EDWARDS, Sebastian

1998 "The Chilean Pension Reform: A Pioneering Program. In: FELDSTEIN, M. (ed.). *Privatizing Social Security.* Chicago: University of Chicago Press, pp. 33-62.

ESCRIVÁ José Luis; Eduardo FUENTES and Alicia GARCÍA-HERRERO

2010 "A. Balance de las reformas de pensiones en Latinoamérica." In: BBVA (ed.). *Las reformas de los sistemas de pensiones en Latinoamérica: avances y temas pendientes.* Madrid: BBVA, pp. 11-39.

FAMA, Eugene and G. William SCHWERT

1977 "Asset Returns and Inflation." In: *Journal of Financial Economics,* vol. 5, no. 2, pp. 115-146.

GESKE, Robert and Richard ROLL

1983 "The Fiscal and Monetary Linkage Between Stock Returns and Inflation." In: *The Journal of Finance,* vol. 38, no. 1, pp. 1-33.

GÓMEZ-HERNÁNDEZ, Denise and Fiona STEWART

2008 "Comparison of Costs and Fees in Countries with Private Defined Contribution Pension Systems." Working paper. Paris: International Organisation of Pension Supervisors.

HABERMAN, Steven and Elena VIGNA

2002 "Optimal Investment Strategies and Risk Measures in Defined Contribution Pension Schemes." In: I*nsurance: Mathematics and Economics,* vol. 31, no. 1, pp. 35-69.

HAN, Nan-wei. and Mao-wei HUNG

2012 "Optimal Asset Allocation for DC Pension Plans Under Inflation. In: *Insurance: Mathematics and Economics,* vol. 51, no. 1, pp. 172-181.

JAMES, Etelle; James SMALHOUT and Dimitri VITTAS

2001 "Administrative Costs and the Organization of Individual Retirement Account Systems: A Comparative Perspective." Policy research working paper No. 2554. New York: World Bank.

KAY, Stephen and Barbara KRITZER

2001 "Social Security in Latin America: Recent Reforms and Challenges. In: *Federal Reserve Bank of Atlanta Economic Review*, vol. 86, no. 1, pp. 41-52.

KELTON, W. David and Averill M. LAW

2000 *Simulation Modeling and Analysis*. Boston: McGraw Hill.

KRITZER, Barbara; Stephen KAY and Tapen SINHA

2011 "Next Generation of Individual Account Pension Reforms in Latin America." In: *Social Security Bulletin*, vol. 71, no. 1, pp. 35-76.

MARTHANS, Juan and José STOK

2013 "Una propuesta para reformar los sistemas privados de pensiones: el caso peruano." Working paper. Piura: Universidad de Piura.

MARTÍNEZ, Óscar and Andrés MURCIA

2008 "Sistema de comisiones de las administradoras de fondos de pensiones en Colombia." Reporte de estabilidad financiera. Bogotá: Banco de la República de Colombia.

MASÍAS, Lorena and Elio SÁNCHEZ

2007 "Competencia y reducción de comisiones en el sistema privado de pensiones: el caso peruano." Working paper No. DT/02/2006. Lima: SBS.

MESA-LAGO, Carmelo

2006 "Private and Public Pension Systems Compared: An Evaluation of the Latin American Experience." In: *Review of Political Economy*, vol. 18, no. 3, pp. 317-334.

MITCHELL, Olivia

1998 "Administrative Costs in Public and Private Retirement Systems." In: FELDSTEIN, M. (ed.). *Privatizing Social Security*. Chicago: University of Chicago Press, pp. 403-456.

MOLOCHE, Guillermo

2012 *Política óptima de inversiones de las AFPs: implicancias del marco regulatorio y los esquemas de comisiones*. Lima: CIES.

MURTHI, Mamta; J. Michael ORSZAG and Peter R. ORSZAG

2001 "Administrative Costs Under a Decentralized Approach to Individual Accounts: Lessons from the United Kingdom." In: HOLZMANN, R. and J. STIGLITZ (eds.). *New Ideas about Old Age Security*. New York: World Bank, pp. 308-335.

POTERBA, James; Joshua RAUH; Steven VENTI and David WISE

2005 "Utility Evaluation of Risk in Retirement Saving Accounts." In: WISE, D. (ed.). *Analyses in the Economics of Aging*. Chicago: University of Chicago Press, pp. 13-58.

QUEISSER, Monika

1998 "Regulation and Supervision of Pension Funds: Principles and Practices." In: *International Social Security Review,* vol. 51, no. 2, pp. 39-55.

SBS

2013 "Anexo técnico 1. Metodología aplicable a los cálculos del aplicativo de comparación de comisiones." Lima: SBS.

SHAH, Hemant

1997 "Towards Better Regulation of Private Pension Funds." Policy research working paper. Nº 1791. New York: World Bank.

SINHA, Tapen

2001 "Analyzing Management Fees of Pension Funds: A Case Study of Mexico." In: *Journal of Actuarial Practice,* no. 9, pp. 5-43.

2000 *Pension Reform in Latin America and its Lessons for International Policymakers,* vol. 23. Dordrecht: Kluwer Academic Publishers.

TAPIA, Waldo and Juan YERMO

2008 "Fees in Individual Account Pension Systems: A Cross-country Comparison." Working Paper on Insurance and Private Pensions No. 27. París: OECD Publishing.

VIGNA, Elena

2014 "On Efficiency of Mean-variance Based Portfolio Selection in Defined Contribution Pension Schemes." In: *Quantitative Finance,* vol. 14, no. 2, pp. 237-258.

WHITEHOUSE, Edward

2001 "Administrative Charges for Funded Pensions: Comparison and Assessment of 13 Countries." In: *Private Pension Systems: Administrative Costs and Reforms, Private Pensions Series.* OECD, no. 2, pp. 85-154.

ZHOU, X. Y. and D. LI

2000 "Continuous-time Mean-variance Portfolio Selection: A Stochastic LQ Framework." In: *Applied Mathematics and Optimization,* vol. 42, no. 1, pp. 19-33.